Basic Seismology 13—Huygens' principle

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According to a story, perhaps apocryphal, the great Dutch scientist Christiaan Huygens (1629–1695) dropped a stone into the canal next to his house, intently observed the circular wavelet that moved out across the water surface, and—like Pythagoras and Plato—intuited that a world of perfection was behind the imperfect visible world and this perfect world was constructed of perfect mathematical and geometric formulations. Of course, the wavelets that Huygens supposedly observed as a boy were never perfectly circular, but his mind held a clear understanding of a perfect circle and it can be argued that, in the spirit of Plato, Huygens spent his life in uncovering the massively important role played by the circle in science.

The Netherlands is a great seafaring nation, and Huygens' first scientific/technological contributions involved improvements to the two most important navigational tools of the 17^{th} century, the telescope and the clock. Huygens, helped by his brother Constanign, designed a telescope that was far superior to contemporary devices and in March 1655 he discovered Saturn's moon Titan. He was also able to explain the curious extension of Saturn, which had intrigued astronomers since Galileo first observed it in 1610: Saturn was encircled by a ring, thin and flat, nowhere touching, inclined to the ecliptic.

In that period, the central problem of navigation was determining longitude. Longitude can, in effect, be measured by time because if the difference in local time at two points is known, the longitudinal distance between them can be computed. However, in the first part of the 17th century, this was not a practical option because the existing mechanical clocks were not sufficiently accurate. Galileo had discovered that a pendulum could be used as a frequency-determining device for a clock and, although many consider him the father of this scientific breakthrough, he never built such a clock. Huygens is the true inventor of the pendulum clock in which the escapement counts the swings and a driving weight provides the push. In effect, the escapement is a feedback regulator that controls the speed of this type of mechanical clock. Huygens produced his first clock in December 1656, and it was much more accurate than contemporary clocks. It is not an overstatement to contend that this was one of the great technological breakthroughs in history. Pendulum clocks were the most accurate clocks in the world for the next 300 years. The invention by Huygens of the first accurate clock can be considered the beginning of the modern world, which is based on science and technology, because it permitted much more sophisticated experiments and detailed measurements.

Huygens was able to construct such a clock because of his investigations of the mathematics of the circle, and this would lead him to additional discoveries that are of major importance in modern geophysics.

Galileo believed that a pendulum is isochronic; in other words, that the period of a pendulum does not depend on the amplitude of its swing. Huygens, via mathematics, found that a pendulum swinging through the arc of the circle is not isochronic. It only appears isochronic when the length of the arc is quite short relative to the length of the pendulum. This property gives a clock with a long pendulum an advantage over a clock with a short pendulum. However, the pendulums of the early clocks were kept short and light to minimize the amount of energy needed to keep them in motion. As a result, the early pendulum clocks had very wide pendulum swings, which decreased their accuracy.

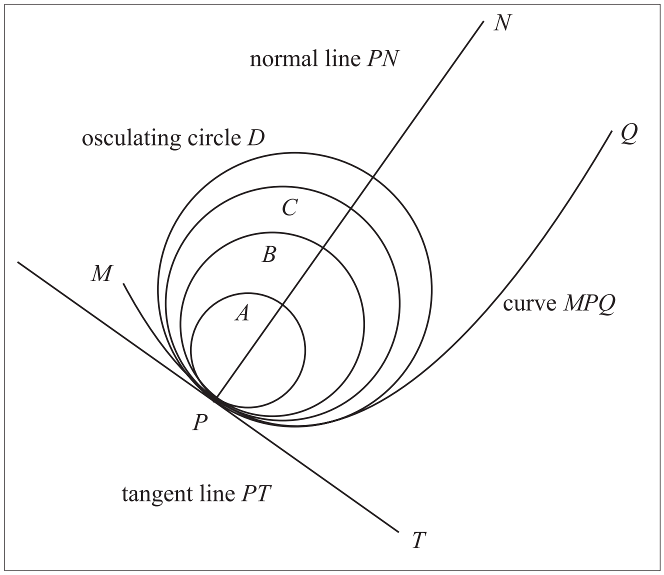

Figure 1.

Huygens, in turning to mathematics, originated what is now known as differential geometry and the idea of curvature, a value that shows how much a curve deviates from a straight line.

Note the curve *MPQ* in Figure 1 and that, at *P*, *PT* is the tangent line and *PN* is the normal line (i.e., the line perpendicular to the tangent). The figure also shows four circles (*A–D*), each tangent to the curve at *P*. Such circles are called tangent circles and each has its center on the normal. There is a unique tangent circle that fits curve *MPQ* at *P* better than any other. This optimum-fitting circle (*D*, in this case) may be described as the circle that "kisses" the curve and it is known as the osculating (derived from the Latin for kissing) circle. The curvature (κ) of *MPQ* at *P* is defined as $\kappa = 1/r$, where r is the radius of the osculating circle (*D*). The sharper the curve at *P*, the larger its curvature. The flatter the curve at *P*, the smaller its curvature—-with the limit being a curve with no curvature, namely a straight line that would coincide with the tangent. In summary, the osculating circle is the circle that touches a curve (on the concave side) and whose radius equals its curvature. Just as the tangent is the line that best approximates a curve at a point, the osculating circle is the circle that best approximates the curve at the point.

Huygens then developed the theory of evolutes. The evolute of a curve is the locus of the centers of the osculating circles of the curve.

In *Moby Dick* (1851), Herman Melville wrote: "I was first indirectly struck by the remarkable fact that all bodies gliding along a cycloid will descend from any point in precisely the same time." The Latinized word for *same time* is *tautochrone* and, in mathematics, the tautochrone problem consists of finding the curve along which a bead placed anywhere on the curve will fall to the bottom of the curve in the same amount of time. The solution is a cycloid, a fact first discovered by Huygens. The cycloid is intimately related

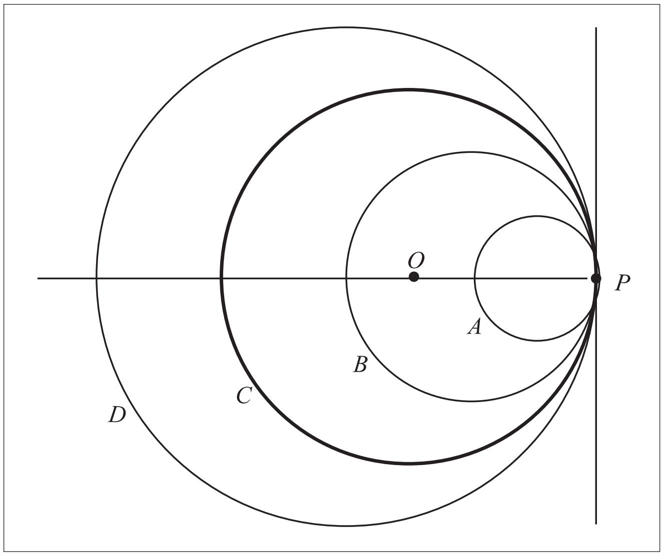

Figure 2.

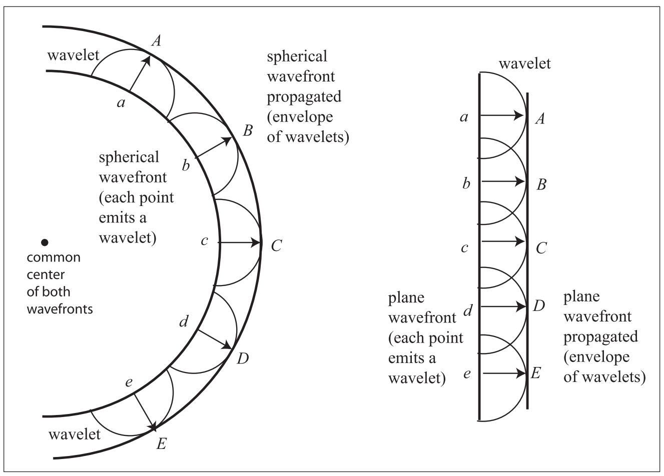

Figure 3.

to the circle; a cycloid is the locus of a point on the rim of a circle rolling along a straight line. In *Horologium Oscillatorium* (1673), Huygens gives a complete mathematical description of an improved pendulum clock and calls such a device a cycloidal clock because its pendulum is forced to swing in an arc of a cycloid. Huygens did this by suspending the pendulum (made up of a bob on a wire string) at the cusp of the evolute of the cycloid. The cycloidal clock was extremely accurate, but unfortunately the movement caused an excessive amount of friction.

Meanwhile, Robert Hooke, another figure with a prominent role in establishing the fundamentals of geophysics, invented the anchor escapement for a pendulum clock. The anchor escapement required a smaller angle of swing than the angle required by the escapements of the early pendulum clocks. Pendulum clocks became so accurate that the cycloidal clock quickly became passé. However, Huygens made one more great contribution to the measurement of time when, in 1675, he built a chronometer that used a balance wheel and a spiral spring instead of a pendulum.

Figure 4.

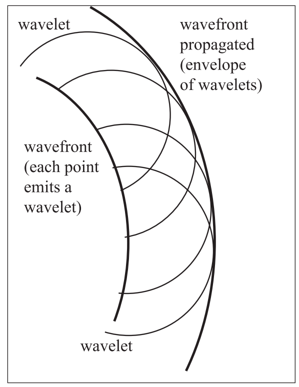

Figure 5.

Balance wheels and spiral springs were the basis for almost all watches until the invention of the quartz crystal oscillator in the twentieth century.

Although the cycloidal clock was of practical use for only a limited time, it is historically important because it can be viewed as the first successful design of an intricate apparatus based on higher mathematics. Heron of Alexandria and Leonardo da Vinci used mechanical principles in order to design their inventions, but the mathematics involved was essentially just Euclidean geometry which dates from about 300 BCE. The introduction of higher mathematics to accomplish mechanical design gives Huygens a claim to being the father of modern technology.

In *Traité de la Lumière* (1690), Huygens made one of the great contributions to theoretical physics when he postulated that light was a wave. Let us attempt to speculate on Huygens' thinking at the time. We can imagine that, in working out the theory of evolutes, Huygens would have asked questions like: What is the evolute of a circle? What is the evolute of a straight line? He would have come up with something like the diagram in Figure 2, which shows that the evolute of circle is the center of the circle. In other words, the osculating circle for circle *C* is circle *C* itself and, similarly, that the evolute of a straight line is a point at infinity. Huygens then would have logically progressed to considering diagrams such as those in Figure 3. The left diagram shows how a spherical wavefront propagates. The right diagram shows how a plane wavefront propagates. In other words, in the spirit of Plato, Huygens used the perfect world of circles to explain the intricacies of wave motion. In 1921

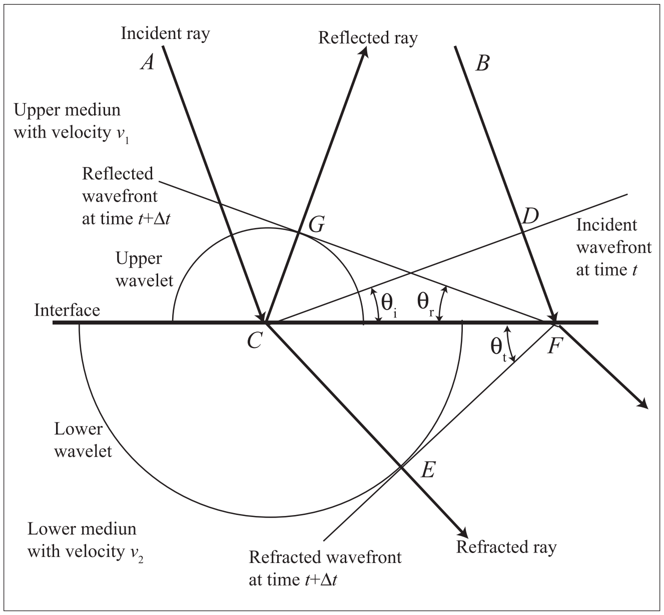

Figure 6.

at Belle Isle, Oklahoma, USA, J. C. Karcher was the first person to record a seismic reflection line. Figure 4 is Karcher's 1921 migration diagram of the Viola interface as given by an envelope of circular arcs. Each time a geophysicist does a prestack depth migration, he or she is using a method based upon the fundamental concept known as Huygens' principle which may be summarized as follows:

> Given a wavefront at a given instant of time, each point on the wavefront emits a spherical wavelet (Figure 5). The wavelets are in phase with the original wavefront and propagate outward with the same speed. The wavelets constructively interfere and their envelope forms a new wavefront. In the same manner the envelope of wavelets from this new wavefront gives the next wavefront.

Huygens imagined that this process repeats itself as the wave propagates. If the medium is homogeneous and isotropic, the spherical wavelets may be constructed with finite radii. On the other hand, if the medium is inhomogeneous, the wavelets will have infinitesimal radii, and the magnitudes of the radii will depend on the wave velocity of the medium at the respective centers of the wavelets. Calculus is needed to deal effectively with these infinitesimals.

Huygens used his wave theory to finally establish the laws of refection and refraction which, particularly the latter, had been sought since ancient times. In Figure 6, a plane interface separates the upper medium from the lower medium. Assume the wave velocity in the lower medium is greater than in the upper medium, i.e. $v_2 > v_1$.

Figure 6 shows a downgoing plane wavefront (*CD*) in the upper medium that is diagonally incident on the interface. As each point on the wavefront arrives at the interface, it behaves according to Huygens' principle and emits two wavelets, one upward and the other downward. There are two envelopes—the envelope that gives the upgoing reflected wavefront and the envelope that gives the downgoing refracted wavefront.

The law of reflection and refraction can be derived by analyzing the part of the incident wavefront that lies between rays *AC* and *BDF* as point *C* contacts the interface (*CF*). Let Δt represent the time increment it takes the wave to travel from *D* to *F* which means that $DF = v_1 \Delta t$. The two wavelets emanating from point *C* are different because the wavelet above the interface is a semicircle with radius $v_1 \Delta t$ and the wavelet below the interface is a semicircle with radius $v_2 \Delta t$. The envelopes at time $t + \Delta t$ are given by the tangent lines *FG* and *FE*.

θ_i between the incident wavefront *CD* and the interface is the angle of incidence. θ_r between the reflected wavefront *FG* and the interface is the angle of reflection. θ_t between the refracted wavefront and the interface is the angle of refraction. These angles are part of three right triangles (*CFD*, *CFG*, and *CFE*) which have the common hypotenuse *CF*. Thus the sines of these three angles have a common denominator, that is

$$\sin\theta_i = \frac{DF}{CF}, \qquad \sin\theta_r = \frac{CG}{CF}, \qquad \sin\theta_t = \frac{CE}{CF},$$

However, $DF = CG = v_1 \Delta t$, and $CE = v_2 \Delta t$ which means that the above equations can be written

$$\sin\theta_i = \frac{v_1\,\Delta t}{CF}, \qquad \sin\theta_r = \frac{v_1\,\Delta t}{CF}, \qquad \sin\theta_t = \frac{v_2\,\Delta t}{CF},$$

Because their sines are equal, it follows that the angle of incidence is equal to the angle of reflection. It also follows that

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{v_1\,\Delta t / CF}{v_2\,\Delta t / CF} = \frac{v_1}{v_2}$$

which is Snell's law—one of the mathematical underpinnings of seismology and a marvelous proof of a relationship that had been sought for centuries.

Thus the wavefront method of Huygens correctly, and with a dramatic simplicity and elegance, generates the laws of reflection and refraction. The work of Huygens on the telescope and the clock would secure his place in the annals of exploration geophysics. His principle of constructing wave motion by the use of secondary wavelets makes Huygens one of the great pioneers of exploration geophysics and, indeed, one of the major figures in the creation of modern science. TLE